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A property of the undominated core for TU games

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Abstract

For a coalitional game with transferble utility, the undominated core is a set of imputations which are not dominated by any other imputations. This set is characterized by reduced game property, individual rationality and a kind of monotonicity.

1 Introduction

In this note we treat solutions for coalitional games with transferable utility. The solutions are the core and the undominated core which were considered in Gillies[3]. We characterize the undominated core, that is, the set of all undominated imputations. The characterization is by axioms, one of which is the reduced game property. In Tadenuma[10], the reduced game by Moulin[7] is used for characterizing the core. We use a variation of the reduced game by Moulin[7]. Llerena/Rafels[6] characterizes the undominated core by another reduced game. The results by Rafels/Tijs[9] and Chang[2] connects the undominated core with the core, and these are effective in our study. For other earlier contributions in this area, see the Reference of [6] and see [8]. For other contributions related to this area, see [1],[4] and [5].

2 Definition of a game

Let $\mathbb{N}$ be the set of natural numbers and let it be the set of *players*. A cooperative game with transferable utility (abbreviated as a *game*) is an ordered pair (N, v), where $N = \{1, \ldots, n\} \subset \mathbb{N}$ is a finite set of n *players* and v, called the *characteristic function*, is a real-valued function on the power set of N, satisfying $v(\emptyset) = 0$. A *coalition* is a subset of N. We denote by Γ the set of all games. For a finite set Z, $|Z|$ denotes the cardinality of Z. For a coalition S, $\mathbb{R}^S$ is the $|S|$-dimensional product space $\mathbb{R}^{|S|}$ with coordinates indexed by players in S. The ith component of $x \in \mathbb{R}^S$ is denoted by x_i. For $S \subseteq N$ and $x \in \mathbb{R}^N$, x_S means the restriction of x to S. We call $x \in \mathbb{R}^N$ a (*payoff*) *vector*. For $S \subseteq N$ and $x \in \mathbb{R}^N$, we define $x(S) = \sum_{i \in S} x_i$ (if $S \neq \emptyset$) and $= 0$ (if $S = \emptyset$). A *pre-imputation* for a game $(N, v) \in \Gamma$ is a vector $x \in \mathbb{R}^N$ that satisfies

$$x(N) = v(N). \tag{1}$$

The set of all pre-imputations for a game $(N, v) \in \Gamma$ is denoted by $X(N, v)$. An *imputation* for a game $(N, v) \in \Gamma$ is a vector $x \in X(N, v)$ that satisfies

$$x_i \geq v(\{i\}), \quad \forall i \in N. \tag{2}$$

$I(N, v)$ is the set of all imputations for a game $(N, v) \in \Gamma$. A *feasible* vector for a game $(N, v) \in \Gamma$ is a vector $x \in \mathbb{R}^N$ that satisfies

$$x(N) \leq v(N). \tag{3}$$

The set of all feasible vectors for a game (N, v) is denoted by $X^*(N, v)$. Let σ be a mapping that associates with every game $(N, v) \in \Gamma'$ a set $\sigma(N, v) \subseteq X^*(N, v)$ where Γ' is a subset of Γ. σ is called a *solution* on Γ'.

Definition 2.1 *A solution σ on Γ' satisfies the Pareto optimality* (PO) if for every game $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq X(N, v)$.

Definition 2.2 *A solution σ on Γ' satisfies the individual rationality* (IR) if for every game $(N, v) \in \Gamma'$, any $x \in \sigma(N, v)$, $x_i \geq v(\{i\})$ for all $i \in N$.

For a game $(N, v) \in \Gamma$, define a game (N, v^-) by[1]

$$v^-(S) = \min\{v(S), v(N) - \sum_{i \in N \setminus S} v(\{i\})\}, \quad \forall S \subseteq N, \tag{4}$$

Definition 2.3 *A solution σ on Γ' satisfies the property I* (PR-I) if for games $(N, v), (N, w) \in \Gamma'$ such that $v^-(S) \geq w^-(S)$ for all $S \subset N$, and $v^-(N) = w^-(N)$, $\sigma(N, v) \subseteq \sigma(N, w)$.

For a game $(N, v) \in \Gamma$, $x \in X^*(N, v)$ and $S \subseteq N$, a *reduced* game is a game $(S, v_S^x) \in \Gamma$. Here S is the player set and v_S^x is the characteristic function which is defined by v, x and S.

Definition 2.4 *A solution σ on Γ' satisfies the reduced game property* (RGP) if for a game $(N, v) \in \Gamma'$, any $x \in \sigma(N, v)$ and any $S \subset N, S \neq \emptyset$, $(S, v_S^x) \in \Gamma'$ and $x_S \in \sigma(S, v_S^x)$.

Definition 2.5 *A solution σ on Γ' satisfies the property II* (PR-II) if for a game $(N, v) \in \Gamma'$, $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$, then $x \in \sigma(N, v)$, where $x_i = v(\{i\})$ for all $i \in N$.

[1]In [6], this game is expressed as (N, v').

3 Core for TU games

In this section the undominated core on Γ is characterized by axioms where the reduced game is defined as follows.

Definition 3.1 *For $(N, v) \in \Gamma$, $x \in \mathbb{R}^N$ and $S \subseteq N$, we define a reduced game $(S, v_S^x) \in \Gamma$ by*

$$\begin{aligned} v_S^x(T) &= \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v(\{i\})\} - x(N \setminus S) \\ &= v^-(T \cup (N \setminus S)) - x(N \setminus S), \quad \forall T \subseteq S, T \neq \emptyset, \\ v_S^x(\emptyset) &= 0. \end{aligned} \tag{5}$$

Remark 3.2 *This reduced game is a variation of the reduced game by Moulin [7]. The latter is used for characterizing the core (See [10]).*

Definition 3.3 *For a game $(N, v) \in \Gamma$ and for $x, y \in X(N, v)$, x dominates y via $S \subset N$ if*

$$\begin{aligned} &x_i > y_i, \forall i \in S, \\ &x(S) \leq v(S). \end{aligned} \tag{6}$$

Definition 3.4 *The undominated core of a game $(N, v) \in \Gamma$, denoted by $DC(N, v)$, is defined by*

$$DC(N, v) = \{x \in I(N, v) : x \text{ is not dominated by any } y \in I(N, v)\}. \tag{7}$$

The core of a game $(N, v) \in \Gamma$, denoted by $C(N, v)$, is defined by

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S), \forall S \subseteq N, S \neq \emptyset\}. \tag{8}$$

The core and the undominated core were considered in Gillies [3]. The following is the main theorem of this paper.

Theorem 3.5 *The undominated core is the only solution on Γ which satisfies* RGP, IR, PR-I, and PR-II.

To prove this theorem, we need 6 lemmas.

Lemma 3.6 *The undominated core on Γ satisfies* RGP.

Proof: It suffices to see when the unmoderated core is nonempty. For $(N, v) \in \Gamma$, suppose $DC(N, v) \neq \emptyset$ and let $x \in DC(N, v)$. Hence $x \in I(N, v)$. For $S \subset N, S \neq \emptyset$, consider (S, v_S^x). By definition,

$$x(S) = v(N) - x(N \setminus S) = v_S^x(S). \tag{9}$$

Claim 3.6A. $x_i \geq v_S^x(\{i\})$ for all $i \in S$.

Proof of Claim 3.6A: If $|S| = 1$, that is, $S = \{i\}$ then $v_{\{i\}}^x(\{i\}) = x_i$ because $x \in I(N, v)$. Let $|S| \geq 2$. Assume $x_i < v_S^x(\{i\})$ for $i \in S$. Then

$$\begin{aligned} x(N \setminus S) + x_i &< x(N \setminus S) + v_S^x(\{i\}) \\ &= v^-(\{i\} \cup (N \setminus S)) \\ &= \min\{v(\{i\} \cup (N \setminus S)), v(N) - \sum_{j \in S \setminus \{i\}} v(\{j\})\} \\ &\leq v(N) - \sum_{j \in S \setminus \{i\}} v(\{j\}). \end{aligned} \tag{10}$$

From this,

$$x(N \setminus S) + x_i + \sum_{j \in S \setminus \{i\}} v(\{j\}) < v(N) = x(N). \tag{11}$$

That is,

$$\sum_{j \in S \setminus \{i\}} v(\{j\}) < x(S \setminus \{i\}). \tag{12}$$

This implies that there exists $j^* \in S \setminus \{i\}$ such that

$$x_{j^*} > v(\{j^*\}). \tag{13}$$

Define $z \in \mathbb{R}^N$ by

$$z_j = \begin{cases} x_j + \varepsilon, & \text{if } j \in \{i\} \cup (N \setminus S); \\ x_{j^*} - \delta, & \text{if } j = j^*; \\ x_j, & \text{otherwise,} \end{cases} \tag{14}$$

where δ and ε are determined so that

$$0 < \delta = \varepsilon|\{i\} \cup (N \setminus S)| < \min\{x_{j^*} - v(\{j^*\}), v^-(\{i\} \cup (N \setminus S)) - x(\{i\} \cup (N \setminus S))\}. \tag{15}$$

Then $z \in I(N, v)$ and z dominates x via $\{i\} \cup (N \setminus S)$ in (N, v). This contradicts $x \in DC(N, v)$. This completes the proof of Claim 3.6A. □

From Claim 3.6A and (9), we see $(S, v_S^x) \in \Gamma_I$ and $x_S \in I(S, v_S^x)$. We shall show $x_S \in DC(S, v_S^x)$. Assume that $y \in I(S, v_S^x)$ dominates x_S via $T \subset S$ in (S, v_S^x). That is,

$$\begin{aligned} y(S) &= v_S^x(S) = x(S), \\ y_i &\geq v_S^x(\{i\}) = v^-(\{i\} \cup (N \setminus S)) - x(N \setminus S), \forall i \in S, \\ y_i &> x_i, \forall i \in T, \\ y(T) &\leq v_S^x(T) = v^-(T \cup (N \setminus S)) - x(N \setminus S). \end{aligned} \tag{16}$$

We let $Q \equiv \{i \in S \setminus T : x_i > v(\{i\})\}$ and $P \equiv \{i \in S \setminus T : x_i = v(\{i\})\}$. By (16),

$$\begin{aligned}
x(T) + x(N \setminus S) &< y(T) + x(N \setminus S) \\
&\leq v^{-}(T \cup (N \setminus S)) \equiv \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v(\{i\})\} \\
&\leq v(N) - \sum_{i \in S \setminus T} v(\{i\}).
\end{aligned} \tag{17}$$

This implies

$$\sum_{i \in S \setminus T} v(\{i\}) < v(N) - x(T) - x(N \setminus S) = x(S \setminus T). \tag{18}$$

Hence there exists $i \in S \setminus T$ such that $x_i > v(\{i\})$. That is, $Q \neq \emptyset$. Define $z \in \mathbb{R}^N$ as follows.

$$z_i = \begin{cases} x_i + \varepsilon_i, & \text{if } i \in N \setminus S; \\ y_i - \delta_i, & \text{if } i \in T; \\ v(\{i\}), & \text{if } i \in P; \\ x_i - \eta_i, & \text{if } i \in Q, \end{cases} \tag{19}$$

where

$$\begin{aligned}
0 < \delta_i &< y_i - x_i, \forall i \in T, \\
\varepsilon_i &> 0, \forall i \in N \setminus S, \\
0 < \eta_i &\leq x_i - v(\{i\}), \forall i \in Q \\
y(T) - x(T) - \delta(T) &+ \varepsilon(N \setminus S) = \eta(Q), \\
\varepsilon(N \setminus S) &\leq \delta(T).
\end{aligned} \tag{20}$$

Indeed, we can find δ_i, ε_i and η_i which satisfy (20) as follows. Since $x(Q) - \sum_{i \in Q} v(\{i\}) > 0$, choose $k \geq 2$ so that

$$0 < \frac{y(T) - x(T)}{k} \leq x(Q) - \sum_{i \in Q} v(\{i\}). \tag{21}$$

Second, choose $\eta_i > 0, \forall i \in Q$ so that

$$\eta(Q) = \frac{y(T) - x(T)}{k} > 0 \text{ and } \eta_i \leq x_i - v(\{i\}), \forall i \in Q. \tag{22}$$

Choose $\delta_i > 0, i \in T$ so that $y_i - x_i - \delta_i < \frac{\eta(Q)}{|T|}$ for all $i \in T$. This implies $y(T) - x(T) - \delta(T) < \eta(Q)$. Finally, determine $\varepsilon_i > 0, i \in N \setminus S$ so that the equality in (20) is satisfied. Then

$$\varepsilon(N \setminus S) - \delta(T) = \eta(Q) - [y(T) - x(T)] = (\frac{1}{k} - 1)[y(T) - x(T)] \leq 0. \tag{23}$$

So (20) is feasible with respect to δ_i, ε_i and η_i. From (19) and (20)

$$\begin{aligned} z(N) &= x(N \setminus S) + \varepsilon(N \setminus S) + y(T) - \delta(T) + \sum_{i \in P} v(\{i\}) + x(Q) - \eta(Q) \\ &= x(N) = v(N). \end{aligned}$$

$$\begin{aligned} z(T \cup (N \setminus S)) &= y(T) + x(N \setminus S) - \delta(T) + \varepsilon(N \setminus S) \\ &\leq v_S^x(T) + x(N \setminus S) - \delta(T) + \varepsilon(N \setminus S) \\ &= \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v(\{i\})\} - \delta(T) + \varepsilon(N \setminus S) \\ &\leq \min\{v(T \cup (N \setminus S)), v(N) - \sum_{i \in S \setminus T} v(\{i\})\} \\ &\leq v(T \cup (N \setminus S)). \end{aligned} \tag{24}$$

From (19) and (20), we see $z_i \geq v(\{i\})$ for all $i \in N$. From this and (24), $z \in I(N,v)$. Consequently, z dominates x via $T \cup (N \setminus S)$ in (N,v), which contradicts $x \in DC(N,v)$. This completes the proof of Lemma 3.6. □

Lemma 3.7 *The undominated core on* Γ *satisfies* IR, PO ,PR-I and PR-II.

Proof: By definition, the undominated core satisfies IR and PO. It is known (Rafels/Tijs(1997)) that for any game (N,v) such that $I(N,v) \neq \emptyset$, $DC(N,v) = C(N,v^-)$. By the definition of the core, $C(N,v^-) \subseteq C(N,w^-)$ for any $(N,v),(N,w)$ such that $v^-(S) \geq w^-(S)$ for all $S \subset N$, and $v^-(N) = w^-(N)$. Since $I(N,v) \neq \emptyset$, we have $I(N,w) \neq \emptyset$, which implies $DC(N,w) = C(Nw^-)$. Hence $DC(N,v) \subseteq DC(N,w)$ and the unmoderated core satisfies PR-I. It satisfies PR-II since any imputation can not dominate itself. □

Lemma 3.8 *If a solution* σ *on* Γ *satisfies* RGP and IR, then it satisfies PO.

Proof: For $(N,v) \in \Gamma$, let $x \in \sigma(N,v)$. By RGP and IR,

$$\begin{aligned} x_i &\geq v_{\{i\}}^x(\{i\}) = \min\{v(\{i\} \cup (N \setminus \{i\})), v(N) - \sum_{j \in \{i\} \setminus \{i\}} v(\{j\})\} - x(N \setminus \{i\}) \\ &= v(N) - x(N \setminus \{i\}). \end{aligned} \tag{25}$$

From this, $x(N) \geq v(N)$. Since $\sigma(N,v) \subseteq X^*(N,v)$, $x(N) \leq v(N)$. Hence we have $x(N) = v(N)$. □

Lemma 3.9 *If a solution* σ *on* Γ *satisfies* RGP, IR, PR-I and PR-II, then $DC(N,v) \subseteq \sigma(N,v)$ for all $(N,v) \in \Gamma$.

Proof: Suppose that a solution σ satisfies RGP, IR and PR-I. For $(N,v) \in \Gamma$, if $DC(N,v) = \emptyset$, then it trivially holds. Suppose $DC(N,v) \neq \emptyset$. So $I(N,v) \neq \emptyset$. Let $x \in DC(N,v) \subseteq I(N,v)$. Since $DC(N,v) = C(N,v^-)$, $x \in C(N,v^-)$. Hence, $x(S) \geq v^-(S)$ for all $S \subseteq N$. Define a game $(N,v_x) \in \Gamma$ by $v_x(S) = x(S)$ for all $S \subseteq N$. Since $x(S) = v_x(S)$ for all $S \subseteq N$ and $(v_x)^- = v_x$, we have $(v_x)^-(S) \geq v^-(S)$ for all $S \subseteq N$ and $(v_x)^-(N) = v^-(N) = v(N)$. By PR-I, $\sigma(N,v_x) \subseteq \sigma(N,v)$. By the assumption and by Lemma 3.8, σ satisfies IR and PO. That is, $\sigma(N,v_x) \subseteq I(N,v_x)$. By PR-II, $x \in \sigma(N,v_x)$. Hence, $x \in \sigma(N,v)$. $\square$

Lemma 3.10 *Suppose that σ on Γ satisfies* RGP and IR. If $v(S) = v^-(S)$ for all $S \subseteq N$ then $\sigma(N,v) \subseteq C(N,v)$.

Proof: Let $x \in \sigma(N,v)$. By RGP, $x_S \in \sigma(S, v_S^x)$ for all $S \subseteq N$. By IR, $x_i \geq v_S^x(\{i\})$ for all $i \in S$. Since $v(S) = v^-(S)$ for all $S \subseteq N$, we have $v(S) \leq v(N) - \sum_{j \in N \setminus S} v(\{j\})$ for all $S \subseteq N$. This implies $v_S^x(\{i\}) = v(\{i\} \cup (N \setminus S)) - x(N \setminus S)$ for all $i \in S$. Hence, $x(N \setminus S) + x_i \geq v(\{i\} \cup (N \setminus S))$ for all $i \in S$. This implies $x(T) \geq v(T)$ for all $T \subseteq N$ since

$$\{\{i\} \cup (N \setminus S) : i \in S, S \subseteq N\} = \{T \subseteq N\}. \tag{26}$$

Hence we have $x \in C(N,v)$. $\square$

Lemma 3.11 *If a solution σ on Γ satisfies* RGP, IR and PR-I, then $\sigma(N,v) \subseteq DC(N,v)$ for all $(N,v) \in \Gamma$.

Proof: Assume $I(N,v) \neq \emptyset$. Since $(v^-)^-(S) = v^-(S)$ for all $S \subseteq N$, by PR-I and Lemma 3.10 we have $\sigma(N,v) = \sigma(N,v^-)$ and $\sigma(N,v^-) \subseteq C(N,v^-)$. Then $C(N,v^-) = DC(N,v)$. Hence $\sigma(N,v) \subseteq DC(N,v)$. Next assume $I(N,v) = \emptyset$. By Lemma 3.8 and IR, $\sigma(N,v) \subseteq I(N,v) = \emptyset$. Hence $\sigma(N,v) = \emptyset \subset DC(N,v)$. $\square$

From Lemmas 3.6 and 3.7, the undominated core satisfies all properties in the statement of the theorem. From Lemma 3.9 and 3.11, a solution on Γ must coincide with the undominated core if it satisfies all properties in the statement of the theorem. This completes the proof of the theorem. $\square$

The next examples show that the properties in Theorem 3.5 are independent.

Example 3.12 *Let $\sigma^1(N,v) = I(N,v)$ for all $(N,v) \in \Gamma$. By definition, σ^1 satisfies* IR, PR-I and PR-II. Let $N = \{1,2,3\}$ and $v(N) = 3, v(13) = v(23) = 2, v(12) = 1$ and $v(i) = 0$ for $i = 1,2,3$. Then $x = (1,2,0) \in I(N,v)$. Let $S = \{1,2\}$. We see $x_{\{1,2\}} \notin I(\{1,2\}, v^x_{\{1,2\}}) = \sigma^2(\{1,2\}, v^x_{\{1,2\}})$ because $v^x_{\{1,2\}}(\{1\}) = 2 > x_1 = 1$. Hence it does not satisfy RGP.

Example 3.13 *Let $\sigma^2(N,v) = \emptyset$ for all $(N,v) \in \Gamma$. Then σ^2 satisfies* IR,PR-I and RGP. But it does not satisfy PR-II.

Example 3.14 *Let* $\sigma^3(N,v) = C(N,v)$ *for all* $(N,v) \in \Gamma$. *By definition,* σ^3 *satisfies* IR and PR-II. Let's see it satisfies RGP. Let $x \in C(N,v)$. Then by definition,$v_S^x(S) = x(S)$ for all $S \subseteq N$. $x(T) = x((N \setminus S) \cup T) - x(N \setminus S) \geq v((N \setminus S) \cup T) - x(N \setminus S) \geq v_S^x(T)$ for all $T \subseteq S$. Hence $x_S \in C(N, v_S^x)$. Next, let's see it does not satisfy PR-I. For $N = \{1,2,3\}$, let $v(i) = w(i) = 0$ for $i = 1,2,3$ and $v(N) = w(N) = 5$. Let $v(12) = w(12) = 2$ and $v(13) = w(13) = 3$. Let $v(23) = 5$ and $w(23) = 6$. Then $C(N,v) = \{(0,2,3)\}$ and $C(N,w) = \emptyset$, while $v^-(S) = w^-(S)$ for all $S \subseteq N$.

Example 3.15 *Let* $\sigma^4(N,v) = \{x \in X^*(N,v) : x_i \leq v(N) - v^-(N \setminus \{i\}), \forall i \in N\}$ *for all* $(N,v) \in \Gamma$. *For sufficiently large* $\varepsilon > 0$, $y_i \equiv v(N) - v^-(N \setminus \{i\}) - \varepsilon < v(\{i\})$ *for some* $i \in N$ *as well as* $y(N) \leq v(N)$, *but* $y \in \sigma^4(N,v)$. *So* $\sigma^4(N,v)$ *does not satisfy* IR. Suppose $v^-(S) \geq w^-(S)$ for all $S \subseteq N$ and $v^-(N) \geq w^-(N)$. Then $v(N) = w(N)$ and $v(N) - v^-(N \setminus \{i\}) \leq w(N) - w^-(N \setminus \{i\})$ for all $i \in N$. This implies $\sigma^4(N,v) \subseteq \sigma^4(N,w)$. Hence σ^4 satisfies PR-I. Next suppose $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. Then $\sigma^4(N,v) = \{x \in X^*(N,v) : x_i \leq v(\{i\}), \forall i \in N\}$, which implies $x \in \sigma^4(N,v)$ where $x_i = v(\{i\})$ for all $i \in N$. Hence σ^4 satisfies PR-II. Next suppose $x \in \sigma^4(N,v)$. Let $S \subseteq N$. Since $x(N) \leq v(N)$, it holds $x(S) \leq v(N) - x(N \setminus S) = v_S^x(S)$.

$$\begin{aligned} v_S^x(S) - (v_S^x)^-(S \setminus \{i\}) &= v_S^x(S) - \min\{v_S^x(S \setminus \{i\}), v_S^x(S) - v_S^x(\{i\})\} \\ &= \max\{v_S^x(S) - v_S^x(S \setminus \{i\}), v_S^x(\{i\})\} \end{aligned} \tag{27}$$

Here

$$\begin{aligned} v_S^x(S) - v_S^x(S \setminus \{i\}) &= v(N) - \min\{v((S \setminus \{i\}) \cup (N \setminus S)), v(N) - v(\{i\})\} \\ &= \max\{v(N) - v((S \setminus \{i\}) \cup (N \setminus S)), v(\{i\})\} \end{aligned} \tag{28}$$

So

$$\begin{aligned} v_S^x(S) - (v_S^x)^-(S \setminus \{i\}) &= \max\{v(N) - v(N \setminus \{i\}), v(\{i\}), v_S^x(\{i\})\} \\ &\geq \max\{v(N) - v(N \setminus \{i\}), v(\{i\})\} \\ &= v(N) - v^-(N \setminus \{i\}) \end{aligned} \tag{29}$$

Hence $x_S \in \sigma^4(S, v_S^x)$. So σ^4 satisfies RGP.

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